24^{th} United States of America Mathematical Olympiad

April 27, 1995
Time Limit: $3\frac{1}{2}$ hours

1. Let p be an odd prime. The sequence $(a_n)_{n\geq 0}$ is defined as follows: $a_0 = 0$, $a_1 = 1, \ldots, a_{p-2} = p-2$ and, for all $n \geq p-1$, a_n is the least positive integer that does not form an arithmetic sequence of length p with any of the preceding terms. Prove that, for all n, a_n is the number obtained by writing n in base $p-1$ and reading the result in base p.

2. A calculator is broken so that the only keys that still work are the sin, cos, tan, $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ buttons. The display initially shows 0. Given any positive rational number q, show that pressing some finite sequence of buttons will yield q. Assume that the calculator does real number calculations with infinite precision. All functions are in terms of radians.

3. Given a nonisosceles, nonright triangle ABC, let O denote the center of its circumscribed circle, and let A_1, B_1, and C_1 be the midpoints of sides BC, CA, and AB, respectively. Point A_2 is located on the ray OA_1 so that ΔOAA_1 is similar to ΔOA_2A. Points B_2 and C_2 on rays OB_1 and OC_1, respectively, are defined similarly. Prove that lines AA_2, BB_2, and CC_2 are concurrent, i.e. these three lines intersect at a point.

4. Suppose $q_0, q_1, q_2, \ldots$ is an infinite sequence of integers satisfying the following two conditions:

   (i) $m - n$ divides $q_m - q_n$ for $m > n \geq 0$,

   (ii) there is a polynomial P such that $|q_n| < P(n)$ for all n.

   Prove that there is a polynomial Q such that $q_n = Q(n)$ for all n.

5. Suppose that in a certain society, each pair of persons can be classified as either *amicable* or *hostile*. We shall say that each member of an amicable pair is a *friend* of the other, and each member of a hostile pair is a *foe* of the other. Suppose that the society has n persons and q amicable pairs, and that for every set of three persons, at least one pair is hostile. Prove that there is at least one member of the society whose foes include $q(1 - 4q/n^2)$ or fewer amicable pairs.