20th USA Mathematical Olympiad

April 23, 1991
Time Limit: $3\frac{1}{2}$ hours

1. In triangle ABC, angle A is twice angle B, angle C is obtuse, and the three side lengths a, b, c are integers. Determine, with proof, the minimum possible perimeter.

2. For any nonempty set S of numbers, let $\sigma(S)$ and $\pi(S)$ denote the sum and product, respectively, of the elements of S. Prove that

$$\sum \frac{\sigma(S)}{\pi(S)} = (n^2 + 2n) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)(n+1),$$

where "Σ" denotes a sum involving all nonempty subsets S of $\{1, 2, 3, \ldots, n\}$.

3. Show that, for any fixed integer $n \geq 1$, the sequence

$$2,\ 2^2,\ 2^{2^2},\ 2^{2^{2^2}}, \ldots \pmod{n}$$

is eventually constant.

[The tower of exponents is defined by $a_1 = 2$, $a_{i+1} = 2^{a_i}$. Also $a_i \pmod{n}$ means the remainder which results from dividing a_i by n.]

4. Let $a = (m^{m+1} + n^{n+1})/(m^m + n^n)$, where m and n are positive integers. Prove that $a^m + a^n \geq m^m + n^n$.

   [You may wish to analyze the ratio $(a^N - N^N)/(a - N)$, for real $a \geq 0$ and integer $N \geq 1$.]

5. Let D be an arbitrary point on side AB of a given triangle ABC, and let E be the interior point where CD intersects the external common tangent to the incircles of triangles ACD and BCD. As D assumes all positions between A and B, prove that the point E traces the arc of a circle.