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**A-1.** Let $S$ be a set of real numbers which is closed under multiplication (that is, if $a$ and $b$ are in $S$, then so is $ab$). Let $T$ and $U$ be disjoint subsets of $S$ whose union is $S$. Given that the product of any *three* (not necessarily distinct) elements of $T$ is in $T$ and that the product of any three elements of $U$ is in $U$, show that at least one of the two subsets $T, U$ is closed under multiplication.

**A-2.** For what pairs $(a, b)$ of positive real numbers does the improper integral

$$\int_b^\infty \left( \sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) dx$$

converge?

**A-3.** The number $d_1 d_2 \ldots d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1 e_2 \ldots e_9$ is such that each of the nine 9-digit numbers formed by replacing just one of the digits $d_i$ in $d_1 d_2 \ldots d_9$ by the corresponding digit $e_i$ ($1 \le i \le 9$) is divisible by 7. The number $f_1 f_2 \ldots f_9$ is related to $e_1 e_2 \ldots e_9$ is the same way: that is, each of the nine numbers formed by replacing one of the $e_i$ by the corresponding $f_i$ is divisible by 7. Show that, for each $i$, $d_i - f_i$ is divisible by 7. [For example, if $d_1 d_2 \ldots d_9 = 199501996$, then $e_6$ may be 2 or 9, since 199502996 and 199509996 are multiples of 7.]

**A-4.** Suppose we have a necklace of $n$ beads. Each bead is labeled with an integer and the sum of all these labels is $n - 1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, \ldots, x_n$ satisfy

$$\sum_{i=1}^k x_i \le k - 1 \qquad \text{for } k = 1, 2, \ldots, n.$$

**A-5.** Let $x_1, x_2, \ldots, x_n$ be differentiable (real-valued) functions of a single variable $t$ which satisfy

$$\frac{dx_1}{dt} = a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n$$
$$\frac{dx_2}{dt} = a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\frac{dx_n}{dt} = a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n$$

for some constants $a_{ij} > 0$. Suppose that for all $i$, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \ldots, x_n$ necessarily linearly dependent?

**A-6.** Suppose that each of $n$ people writes down the numbers 1, 2, 3 in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums $a, b, c$ of the resulting matrix be rearranged (if necessary) so that $a \le b \le c$. Show that for some $n \ge 1995$, it is at least four times as likely that both $b = a + 1$ and $c = a + 2$ as that $a = b = c$.

**B-1.** For a partition $\pi$ of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing $x$. Prove that for any two partitions $\pi$ and $\pi'$, there are two distinct numbers $x$ and $y$ in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set $S$ is a collection of disjoint subsets (parts) whose union is $S$.]

**B-2.** An ellipse, whose semi-axes have lengths $a$ and $b$, rolls without slipping on the curve $y = c \sin(x/a)$. How are $a, b, c$ related, given that the ellipse completes one revolution when it traverses one period of the curve?

**B-3.** To each positive integer with $n^2$ decimal digits we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det \begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50$. Find, as a function of $n$, the sum of all the determinants associated with $n^2$-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

**B-4.** Evaluate

$$\sqrt[8]{2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \dots}}}.$$

Express your answer in the form $(a + b\sqrt{c})/d$, where $a, b, c, d$ are integers.

**B-5.** A game starts with four heaps of beans, containing 3,4,5 and 6 beans. The two players move alternately. A move consists of taking *either*

    a. one bean from a heap, provided at least two beans are left behind in that heap, or

    b. a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

**B-6.** For a positive real number $\alpha$, define

$$S(\alpha) = \{\lfloor n\alpha \rfloor : n = 1, 2, 3, \dots\}.$$

Prove that $\{1, 2, 3, \dots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$ and $S(\gamma)$. [As usual, $\lfloor x \rfloor$ is the greatest integer $\leq x$.]