38^{th} International Mathematical Olympiad

Mar del Plata, Argentina

Day I

July 24, 1997

1. In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white (as on a chessboard).

   For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along edges of the squares.

   Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let

   $$f(m,n) = |S_1 - S_2|.$$

   (a) Calculate $f(m,n)$ for all positive integers m and n which are either both even or both odd.

   (b) Prove that $f(m,n) \leq \frac{1}{2}\max\{m,n\}$ for all m and n.

   (c) Show that there is no constant C such that $f(m,n) < C$ for all m and n.

2. The angle at A is the smallest angle of triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T. Show that

   $$AU = TB + TC.$$

3. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions

   $$|x_1 + x_2 + \cdots + x_n| = 1$$

   and

   $$|x_i| \leq \frac{n+1}{2} \qquad i = 1, 2, \ldots, n.$$

   Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

   $$|y_1 + 2y_2 + \cdots + ny_n| \leq \frac{n+1}{2}.$$

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July 25, 1997

4. An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver* matrix if, for each $i = 1, 2, \ldots, n$, the ith row and the ith column together contain all elements of S. Show that

   (a) there is no silver matrix for $n = 1997$;

   (b) silver matrices exist for infinitely many values of n.

5. Find all pairs (a, b) of integers $a, b \geq 1$ that satisfy the equation

$$a^{b^2} = b^a.$$

6. For each positive integer n , let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways:

$$4; 2+2; 2+1+1; 1+1+1+1.$$

   Prove that, for any integer $n \geq 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$