34nd International Mathematical Olympiad

First Day — July 18, 1993
Time Limit: $4\frac{1}{2}$ hours

1. Let $f(x) = x^n + 5x^{n-1} + 3$, where $n > 1$ is an integer. Prove that $f(x)$ cannot be expressed as the product of two nonconstant polynomials with integer coefficients.

2. Let D be a point inside acute triangle ABC such that $\angle ADB = \angle ACB + \pi/2$ and $AC \cdot BD = AD \cdot BC$.

   (a) Calculate the ratio $(AB \cdot CD)/(AC \cdot BD)$.

   (b) Prove that the tangents at C to the circumcircles of $\triangle ACD$ and $\triangle BCD$ are perpendicular.

3. On an infinite chessboard, a game is played as follows. At the start, n^2 pieces are arranged on the chessboard in an n by n block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is removed.

   Find those values of n for which the game can end with only one piece remaining on the board.

Second Day — July 19, 1993
Time Limit: $4\frac{1}{2}$ hours

4. For three points P, Q, R in the plane, we define $m(PQR)$ as the minimum length of the three altitudes of $\triangle PQR$. (If the points are collinear, we set $m(PQR) = 0$.)

   Prove that for points A, B, C, X in the plane,

$$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

5. Does there exist a function $f : \mathbf{N} \to \mathbf{N}$ such that $f(1) = 2$, $f(f(n)) = f(n) + n$ for all $n \in \mathbf{N}$, and $f(n) < f(n+1)$ for all $n \in \mathbf{N}$?

6. There are n lamps $L_0, \ldots, L_{n-1}$ in a circle ($n > 1$), where we denote $L_{n+k} = L_k$. (A lamp at all times is either on or off.) Perform steps $s_0, s_1, \ldots$ as follows: at step s_i, if L_{i-1} is lit, switch L_i from on to off or vice versa, otherwise do nothing. Initially all lamps are on. Show that:

   (a) There is a positive integer $M(n)$ such that after $M(n)$ steps all the lamps are on again;

   (b) If $n = 2^k$, we can take $M(n) = n^2 - 1$;

   (c) If $n = 2^k + 1$, we can take $M(n) = n^2 - n + 1$.