Fluids

Core to a fluids problem are the following ideas:

1. A fluid element is an infinitesimal "block" of fluid that has the same properties as macroscopic fluid body.
2. Pascal's Principal: The pressure applied to a stationary fluid increases the pressure throughout equally.
    (a) The pressure on all sides of a fluid element is the same.
    (b) The pressure is nonzero only perpendicular to the face of any fluid element.
3. $P = \rho g h$
4. $\frac{dP}{dy} = -\rho g$, where ρ is the density of the fluid.
5.
6.
7.

Some userful facts you should eventually memorize (if you haven't already):

1. $Pressure = \frac{Force}{Area} \left([Pa] = \frac{[N]}{[m^2]} \right)$, where Pa stands for Pascal, which is the SI unit for pressure.
2. $\rho_{water} = 1000\ kg/m^3 = 1\ g/cm^3$
3. $\rho_{air} = 1.29\ kg/m^3$
4. $P_0 = 1\ atm = 101.3 kPa$
5. Specific gravity is defined as: $\frac{\rho_{substance}}{\rho_{water}}$

1 Pascal's Principal

Pascal's Principal states that *the pressure applied to a stationary fluid increases the pressure throughout equally.* This means that if you apply a pressure of 2 atmospheres on one end of a fluid-filled long pipe, the pressure on the other end (and in the middle) will increase by 2 atmospheres.

Define a fluid element to be a "block" of fluid big enough to still act like the body of fluid it is a part of, but small enough that it can be considered a fundamental "piece" of the fluid body.

A pressure applied to one end of the fluid-filled pipe will act on all the fluid elements in the pipe. Action-reaction thereby applies the same pressure to the other side (and all elements in the middle). Example: if it is known that a pressure of 2 atm acts on the top surface of a fluid element, a pressure of 2 atm also acts on the left surface, the back surface, the right surface, the front surface, and the bottom surface.

From Pascal's Principal, one can deduce that the pressure on all sides of a (stationary) fluid element must be equal. If this were not the case, the fluid would be moving — for example, if the pressure acting on the left side is greater than that acting on the right side, then the fluid would move left. Because the pressure applied increases the pressure throughout equally, the fluid elements must have pressures that cancel in each direction; therefore, the pressure must be equal on all sides.

Also from Pascal's Principal, one can deduce that the pressure is nonzero only perpendicular to the face of a fluid element. (The force acts normal to the face.) If this were not the case, then the fluid would flow.

2 $P = \rho gh$

By definition, $P = \frac{F}{A} = \frac{mg}{A}$ (F=force; A=area). The density of the fluid is $\rho = \frac{m}{V} \Rightarrow m = \rho V$ (m=mass; V=volume). Thus, $P = \rho Vg/A$. The volume of an element is the area of the base times its height, $V = Ah$. Therefore, $P = \rho gh$. Note that because V is the volume of fluid above the point where the pressure is taken, the height h is the height above the point where pressure is to be determined.

2.1 Example:

: Find the pressure at the bottom of a $50m$ tall drain pipe, assuming that there is a constant supply of water being drained. Assume that atmospheric pressure is the same at the bottom and top of the well.

$P = \rho_{water} gh = 1000 \cdot 9.8 \cdot 50Pa$

2.1.1 What if the bottom of the pipe is closed?

: The pressure acting downwards at the top surface of a fluid element is $P_0 + \rho_{water} gh$. This is also equal to the pressure acting upwards on the bottom surface of a fluid element.

3 $\frac{dP}{dy} = -\rho g$, where ρ is the density of the fluid.

Place the datum at the bottom of a beaker of fluid. A pressure P acts downwards on the upper surface of the fluid element. A pressure $P + dP$ acts upwards on the bottom surface of the fluid element. Each face has area A. A pressure of ρgdy acts downwards on the fluid element. Thus, the force balance implies:

$$\begin{aligned} (P)A - (P + dP)A - \rho gdyA &= 0 \quad (1) \\ P - P + dP - \rho gdy &= 0 \quad (2) \end{aligned}$$

$$\begin{aligned} -dP - \rho g dy &= 0 \qquad (3) \\ \frac{dP}{dy} &= -\rho g \qquad (4) \end{aligned}$$

3.1 Example

Suppose it is known that the fluid density varies proportional the pressure. $\rho = kP$

(Assume that $\rho g dy$ still applies to the pressure contributed by the fluid element.)

$$\begin{aligned} \frac{dP}{dy} &= -\rho g \qquad (5) \\ \frac{dP}{dy} &= -kPg \qquad (6) \\ \frac{dP}{P} &= -kgdy \qquad (7) \\ ln(P_2) - ln(P_1) = ln(\frac{P_2}{P_1}) &= -kg(y_2 - y_1) \qquad (8) \\ \frac{P_2}{P_1} &= \exp(-kg\Delta y) \qquad (9) \\ \frac{P_2}{P_1} &= \exp(-\frac{\rho_0}{P_0} g\Delta y) \qquad (10) \end{aligned}$$

In the last step, the following equality is used: $\frac{P_0}{P} = \frac{\rho_0}{\rho} \Rightarrow k = \frac{\rho}{P} = \frac{\rho_0}{P_0} = \frac{1.29}{101.3E3}$, where P_0 and ρ_0 refer to the atmospheric pressure and density at sea level, respectively.