Capacitors

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1 Bare Basics

The simplest capacitor is made of two parallel conducting plates seperated by a thin layer of insulation. When the capacitor is charged, a voltage difference exists across the two plates.[1] The charge (Q) that accumulates on either plate[2] is *proportional to the potential difference (V) between the plates.*

$$Q = CV \tag{1}$$

The constant of proportionality is known as the capacitance (C). The units for C are $[charge] \, / \, [voltage] = Coulomb/Volt$, which is defined as the Farad (F). Capacitors are identified by their capacitance (and geometry).

2 Determination of Capacitance

One can determine an intrinsic formula for finding the capacitance of a capacitor. The recipe goes like:

1. Find the (electric) field ($\vec{E}$) between the plates of the capacitor.
2. Find the voltage, viz., $V = -\int \vec{E} \cdot d\vec{l}$.
3. Use (1) to find $C = Q/V$. Q should cancel out once you plug in V from step 2.

Three well-done dishes await you below in the guise of (gasps) examples. Formulae for the most commonly found capacitors will be determined below. In addition, it will be shown that in the limit of small seperations, both the cylindrical and spherical capacitors reduce to that of a parallel plate capacitor.

[1] This potential difference induces a field in between the plates. One plate is thus positively charged, and the other is negatively charged. The positive plate "cancels" the negative plate completely, and the net charge of both plates is zero. However, when the voltage source is disconnected, the capacitor still stores the charge that accumulated due to the potential from the voltage source. The charge stored is the amount of charge that accumulated on either plate. (Sign conventions differ depending on which segment of the circuit one is considering, but it is usually the absolute value of the charge that is of importance. More on this later.)

[2] Rather, the charge that the plate acquires.

The bulk of this section exists in the examples below. The examples will also review Gauss' Law. Recall the famous ($\vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon}$). These steps determine formulae for capacitance that are independent of the charge accumulated, but can be computed from just the geometry of the capacitor.[3]

2.1 Parallel Plate Cap's

1. **Electric field**

   Let the area parallel to the surface be A. The normal vector of this area is parallel to the electric field. The flux is thus just $EA = Q/\epsilon$. Hence, the field is $\vec{E} = \frac{Q}{A\epsilon} = \frac{\sigma}{\epsilon}$, where $\sigma = \frac{Q}{A}$.

2. **Voltage**

   Plug in E from step 1 into the voltage equation: $V = -\int_0^d \frac{\sigma}{\epsilon}\hat{z} \cdot dl(-\hat{z}) = \frac{\sigma d}{\epsilon} = \frac{Qd}{A\epsilon}$

3. **Capacitance**

   $C = Q/V = \frac{Q}{\frac{Qd}{A\epsilon}} = \frac{A\epsilon}{d}$

Since parallel-plate cap's are ubiquitous (plus or minus a few exceptions), the above formula is worth repeating.[4]

$$C_{ParaPlates} = \frac{A\epsilon}{d} \tag{2}$$

The formula makes sense.[5] If you make the plates larger in area, the capacitor will have a greater capacity to store charge, thus have a greater capacitance. If you increase the distance between the plates, the field induced will get weaker, by Coulomb's law, and thus less charge will accumulate on the plates.[6]

[3] Experimentally, one can also determine the capacitance from knowing/measuring the voltage and charge. Aside from the toils of life, labwork is trivial.

[4] The beauty of typing this in lieu of *handwriting* this is that all I have to do above is copy and paste. But, if you're taking notes off me, the more power to the pain on your writing caluses!

[5] Note that it is independent of the charge, even though capacitance tends to be taken intuitively as the amount of charge capacity of a capacitor. The idea is that you can predetermine the charge capacity by knowing just the plate area and distance between the paltes. (Charge cancels out in the derivation. But, charge accumulation is what makes a capacitor useful.)

[6] If you worry about causality, here's how it might go: You put a distance between the plates and charge them up with a potential source. Charge gathers on one plate. Charge of the opposite sign is induced on the other plate. A field is induced between them. Decharge them. Take the same two plates and put a greater distance between them. Hook them up to a voltage source. Charge parade blockage on either plate. *Less* charge is induced on the other plate, and thus there is less of a field. Depending on whether you see charge as more fundamental than field, it is either the charge or the field that produces the change in capacitance.

2.2 Co-Ax Cap's

Co-Ax capacitors, a.k.a. coaxial capacitors, consist of two concentric cylinders sharing the same axis. A voltage source is hooked up to either plate and the other plate closes the circuit by reconnecting to the voltage source.

Our co-ax cap will have an inner radius of r_s and an outer radius of r_b. Note that r_s is *s*maller than r_b, which is *b*igger than r_s.

1. **Electric field**

   Recall the area of a cylinder of radius R and length L (excluding the two ends/bases): $2\pi RL$. The normal vector of the area is always perpendicular to the round surfaces of the cylinder. This is either parallel or antiparallel to the field.[7] The flux is $E(2\pi rL) = \frac{Q}{\epsilon}$. Thus the field is $E = \frac{Q}{2\pi rL\epsilon}$.[8]

2. **Voltage** (The outer surface is taken as the lower potential. The inner surface is the one that is positively charged.)

   Plug in E from step 1 into the voltage equation:
   $V = -\int_{r_b}^{r_s} \frac{Q}{2\pi rL\epsilon}\hat{r} \cdot dl(-\hat{r}) = \frac{Q}{2\pi L\epsilon} \ln(r_b/r_s)$.

3. **Capacitance**
   $C = Q/V = \frac{Q}{\frac{Q}{2\pi L\epsilon}\ln(r_b/r_s)} = \frac{2\pi L\epsilon}{\ln(r_b/r_s)}$

$$C = \frac{2\pi L\epsilon}{\ln(r_b/r_s)} \tag{3}$$

You should probably know this equation by heart, too.

2.2.1 Limiting Case $r_b - r_s \ll r_b$

$r_b - r_s \ll r_b \Rightarrow y = (1 - \frac{r_s}{r_b}) \ll 1$.

Recall the Taylor expansion of $\ln(1-y) \approx -y - y^2/2$.

$C = -\frac{2\pi L\epsilon}{\ln(r_s/r_b)} = -\frac{2\pi L\epsilon}{\ln(1-y)} \approx -\frac{2\pi L\epsilon}{-y} = \frac{2\pi L\epsilon}{y} = \frac{2\pi L\epsilon}{1-r_s/r_b} = \frac{\epsilon 2\pi L r_b}{r_b - r_s} = \frac{\epsilon A}{d}$.
Where in the last step, the distance between the plates is the distance between the two radii $d = r_b - r_s$, and the area is taken to be based on the outer radius $A = 2\pi r_b L$. Thus, in the limit that the seperation between the inner and outer radius is small (i.e., much smaller than the outer radius), *the capacitance of a coax capacitor reduces to that of a parallel plate capacitor.*

[7] The sign cancels out, since if it's antiparallel to the field, it must be on the "plate" with the negative charge.

[8] r and L are used instead of R and L because the field varies depending on the radius of the Gaussian surface.

2.3 Spherical Cap's

A spherical capacitor consists of two concentric spheres sharing the same origin. A voltage source is hooked up to either spherical surface and the other surface closes the circuit by reconnecting to the voltage source.

Our spherical capacitor will have an outer radius of r_b (*b*igger) and an inner radius of r_s (*s*maller).

1. **Electric field**

   Recall the area of a sphere of radius R: $4\pi R^2$. The normal vector of the area is always perpendicular to the surfaces of the sphere. This is either parallel or antiparallel to the field.[9] The flux is $E(4\pi r^2) = \frac{Q}{\epsilon}$. Thus the field is $E = \frac{Q}{4\pi r^2 \epsilon}$.

2. **Voltage**

   Plug in E from step 1 into the voltage equation:
   $V = -\int_{r_b}^{r_s} \frac{Q}{4\pi r^2 \epsilon} \hat{r} \cdot dl(\hat{r}) = \frac{Q}{4\pi\epsilon}\left(\frac{1}{r_s} - \frac{1}{r_b}\right) = \frac{Q}{4\pi\epsilon}\left(\frac{r_b - r_s}{r_b r_s}\right)$.

3. **Capacitance**

   $C = Q/V = \frac{Q}{\frac{Q}{4\pi\epsilon}\left(\frac{r_b - r_s}{r_b r_s}\right)} = \frac{4\pi\epsilon r_b r_s}{r_b - r_s}$

2.3.1 Limiting Case ($r_b - r_s \ll r_b$)

The distance between the two spherical surfaces is the distance between the conducting plates: $r_b - r_s = d$. This regime implies that $r_b \approx r_s$. Thus $r_b r_s \approx r_b^2$, and

$C = \frac{4\pi\epsilon r_b r_s}{r_b - r_s} \approx \frac{4\pi r_b^2 \epsilon}{d} = \frac{A\epsilon}{d}$. This is the the parallel plate (2) capacitance, again.

3 Parallel and Series

More than one capacitor can be connected together to form any *equivalent capacitance* you want. Capacitors are really only useful when there is a voltage source supplying a flow of charge towards the capacitor. A closed path is a closed loop of conductors; a loop is closed if charge can flow back (on a one-way wire) to their starting position. Usually, the starting/ending point is a voltage source or a battery.

There are two general ways to combine capacitors. The idea is that the most complicated capacitor circuit can be broken down into its constituent combinations: capacitors arranged in parallel and series. Each arrangement of capacitors can be broken down into the simplest case by reducing all the

[9] The sign cancels out, since if it's antiparallel to the field, it must be on the "plate" with the negative charge.

capacitances to an *equivalent capacitance* (C_{eq}). The holistic formula for the total charge (Q) of any capacitor circuit of voltage V_0 is

$$Q = C_{eq} V_0. \tag{4}$$

3.1 *Capacitors in parallel share the same voltage.*

Suppose you have just two capacitors, say C_1 and C_2, connected together in parallel to a voltage source V_0. By definition, you get this equation: $V_1 = V_2 = V_0$. The total charge is $Q = Q_1 + Q_2 = C_1 V_1 + C_2 V_2 = (C_1 + C_2)V_0 = C_{eq} V_0$. The equivalent capacitance for *two* capacitors put together in parallel is $C_{eq} = C_1 + C_2$.

In general, the equivalent capacitance of n capacitors in parallel is:

$$C_{eq}(n) = \sum_{i=1}^{n} C_i \tag{5}$$

If you have n capacitors all of the same capacitance C_0, then $C_{eq} = nC_0$.

3.2 *Capacitors in series share the same charge.*

Suppose you have just two capacitors, say C_1 and C_2, connected together in series to a voltage source V_0. By definition, you get this equation: $Q_1 = Q_2 = Q$. The total charge is equal to the charge on either capacitor.[10] The total voltage is the sum of the individual voltages $V_0 = V_1 + V_2 = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = Q\left(\frac{C_1+C_2}{C_1 C_2}\right) = \frac{Q}{C_{eq}}$. The equivalent capacitance of two capacitors in series is: $C_{eq} = \frac{1}{C_1} + \frac{1}{C_2} = (\frac{C_1+C_2}{C_1 C_2})^{-1} = \frac{C_1 C_2}{C_1+C_2}$

In general, the equivalent capacitance of n capacitors in series is:

$$C_{eq}(n) = \left(\sum_{i=1}^{n} \frac{1}{C_i}\right)^{-1} \tag{6}$$

If you have n capacitors all of the same capacitance C_0, then $C_{eq} = \frac{C_0}{n}$

Note the difference in strategy for determining the equivalent capacitor in series and parallel capacitors: In parallel, we take the sum of the *charges*. In series, we take the sum of the *voltages*. After that, we then plug in the proverbial (1). And, that's that.

[10] You can see this by "canceling out" the middle two plates. Take just the first and last plate, and you essentially get just one capacitor—in this case, the equivalent capacitance.

4 More Examples

4.1

5 Biblio

Giancoli. *Physics for Scientists and Engineers*. 3rd Ed.